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ANALYSIS OF IMPACT RESPONSE IN COMPOSITE PLATES

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Abstract—The problem of impact is of considerable interest in laminated composite materials. Although important contributions have been made in understanding the impact problem through numerical solutions, an analytical solution has not been available for the problem of impact of laminated plates. The present work gives an analytical solution to this problem, based on the usual Fourier series expansion for simply-supported plates, combined with Laplace transform techniques for the impact problem solution.

INTRODUCTION

The analysis of laminated composite plates subject to impact loads has been of much interest in recent years, because of the sensitivity of advanced composite materials to impact damage. Despite considerable progress in the analysis of laminated orthotropic plates under dynamic loads, an analytical solution to the impact problem is not available in the literature. Previous investigators have presented analytical solutions for the response of composite plates under time dependent loads, and numerical solutions for the problem of impact of composite plates by either rigid or elastic impactors. Analyses are available that include the nonlinear contact indentation at the point of impact by a foreign object as well as the plate dynamics, but perform the time integration numerically. However, in recent work Christoforou and Swanson (1988) have shown that it is possible to obtain an analytical solution to the impact problem by using Laplace transform techniques, if the nonlinear contact indentation is approximated by a linearized version. The present work shows how this approach can be applied to orthotropic laminated plates.

The analysis of static and dynamic loadings of laminated composite plates has seen considerable development recently. Lekhnitskii (1968), Whitney and Leissa (1970) and Jones (1975), have presented analyses with infinite transverse rigidity, while Whitney and Pagano (1970) have presented a theory which includes transverse shear deformations. The dynamic problems considered involved vibration of plates. Sun and Chattopadhyay (1975) and Dobyns (1981) used the plate equations developed by Whitney and Pagano (1970) to analyze a simply-supported orthotropic plate subject to center impact. Dobyns assumed that the lateral force history was known, while Sun and Chattopadhyay noted that in impact by a foreign body, the force history must be computed as part of the problem. Thus Sun and Chattopadhyay integrated the equations of motion numerically. Birman and Bert (1987) obtained a closed-form solution for laminated angle-ply simply-supported plates subject to blast loading, which again considers that the force history is known.

As mentioned above, a closed-form solution of the impact problem has not been obtained previously. Because the inclusion of contact deformation effects yields a nonlinear term in the integral equation, it is unlikely that a closed-form solution will be available for this problem, which thus has to be solved numerically as discussed above. In the present paper the nonlinear contact stiffness is replaced by a linearized stiffness, to provide an estimate of the additional compliance due to contact area deformation effects. Thus, bounds can be placed on the response with the linearized approach while still using an analytic solution. The solution procedure is similar to that presented earlier for the impact response of composite cylinders (Christoforou and Swanson, 1988).

In the following, the method of solution is described. Comparisons with previous

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results in the literature are given, and illustrations of the effect of variables in the impact problem are presented.

ANALYSIS

The plate equations of motion developed by Whitney and Pagano (1970) reduced to specially orthotropic form ($B_{ij} = 0$, $A_{16} = A_{26} = D_{16} = D_{26} = 0$), are (Dobyns, 1981)

$$D_{11}\frac{\partial^2\psi_x}{\partial x^2}+D_{66}\frac{\partial^2\psi_x}{\partial y^2}+(D_{12}+D_{66})\frac{\partial^2\psi_y}{\partial x\,\partial y}-kA_{55}\left(\psi_x+\frac{\partial w}{\partial x}\right)=\frac{\rho h^3}{12}\frac{\partial^2\psi_x}{\partial t^2}$$

$$(D_{12}+D_{66})\frac{\partial^2\psi_x}{\partial x\,\partial y}+D_{66}\frac{\partial^2\psi_y}{\partial x^2}+D_{22}\frac{\partial^2\psi_y}{\partial y^2}-kA_{44}\left(\psi_y+\frac{\partial w}{\partial y}\right)=\frac{\rho h^3}{12}\frac{\partial^2\psi_y}{\partial t^2}$$

$$kA_{55}\left(\frac{\partial\psi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+kA_{44}\left(\frac{\partial\psi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+q(x,y,t)=\rho h\frac{\partial^2 w}{\partial t^2}, \qquad (1)$$

where D_{ij} and A_{ij} are the stiffnesses, as defined by Whitney and Pagano (1970), h is the plate thickness, t is time, ρ is material density, w is the plate displacement in the z direction at the plate midplane, ψ_x and ψ_y are the shear rotations in the x and y directions and k is a shear correction factor introduced by Mindlin (1951) and is popularly taken to be $\pi^2/12$.

Unlike the case with orthotropic shells, taking B_{ij} equal to zero decouples the plate equations of motion into two independent groups, of which only the group concerned with transverse displacement is considered here.

This paper is concerned with a simply-supported rectangular plate of uniform thickness with dimensions a and b for which the boundary conditions are given by:

$$w=\frac{\partial\psi_x}{\partial x}=0 \quad \text{at } x=0,a$$

$$w=\frac{\partial\psi_y}{\partial y}=0 \quad \text{at } y=0,b. \qquad (2)$$

SOLUTION OF THE DYNAMICS PROBLEM

The solution is based on expansions of the loads, displacements and rotations in Fourier series which satisfy the end boundary conditions of simple support. Each expression is assumed to be separable into a function of time and a function of position. Furthermore, following Bert and Birman (1987), by neglecting in-plane and rotary inertia the problem becomes a second-order ordinary differential equation in time for the Fourier coefficients of the lateral deflection. In the case of impact, the impact force is computed from the deceleration of the impactor mass. This involves the equilibrium equation between the impactor and the plate during contact.

For a given dynamic load, solutions of the governing eqns (1) that satisfy the boundary conditions, eqns (2), are given by:

$$\psi_x=\sum_m\sum_n A_{mn}(t)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b},$$

$$\psi_y=\sum_m\sum_n B_{mn}(t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b},$$

$$w=\sum_m\sum_n W_{mn}(t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \qquad (3)$$

with the load function represented by

$$q(x,y,t) = \sum_m \sum_n Q_{mn}(t) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}. \tag{4}$$

Equations (3) and (4) are the Fourier series representation of the rotations, lateral displacement and load. The terms of the Fourier series representation for a uniform load over the rectangular area u, v with center at ζ, η as shown in Fig. 1 (needed for the contact loading, as will be discussed subsequently) are (Dobyns, 1981)

$$Q_{mn}(t) = \frac{16F(t)}{\pi^2 mnuv} \sin\frac{m\pi\zeta}{a} \sin\frac{n\pi\eta}{b} \sin\frac{m\pi u}{2a} \sin\frac{n\pi v}{2b}. \tag{5}$$

In the present study, following the results of Bert and Chen (1978), rotary inertia effects are neglected. Thus, substituting eqns (3) and (4) into eqns (1) results in independent sets of three equations for each set of modal parameters m and n:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{Bmatrix} A_{mn}(t) \\ B_{mn}(t) \\ W_{mn}(t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ Q_{mn}(t) - \rho h \ddot{W}_{mn}(t) \end{Bmatrix} \tag{6}$$

where the elements of the symmetric matrix C_{ij} are

$$C_{11} = D_{11}\left(\frac{m\pi}{a}\right)^2 + D_{66}\left(\frac{n\pi}{b}\right)^2 + kA_{55}$$

$$C_{12} = (D_{12} + D_{66})\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)$$

$$C_{13} = kA_{55}\left(\frac{m\pi}{a}\right)$$

$$C_{22} = D_{66}\left(\frac{m\pi}{a}\right)^2 + D_{22}\left(\frac{n\pi}{b}\right)^2 + kA_{44}$$

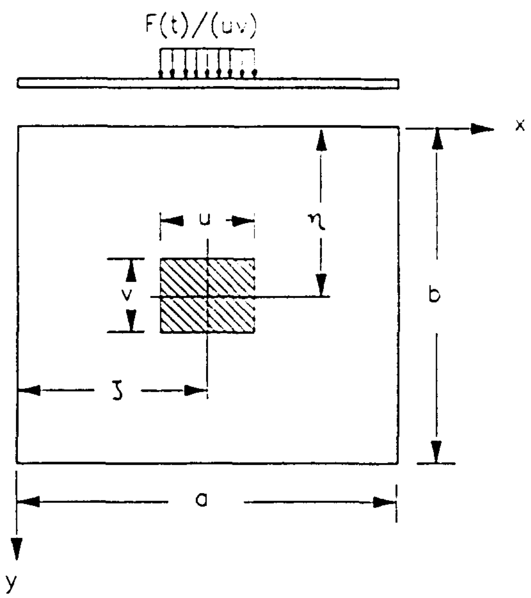

Fig. 1. Illustration of plate geometry.

$$C_{23} = kA_{44}\left(\frac{n\pi}{b}\right)$$

$$C_{33} = kA_{55}\left(\frac{m\pi}{a}\right)^2 + kA_{44}\left(\frac{n\pi}{b}\right)^2. \tag{7}$$

Following Bert and Birman (1987), eqns (6) can be reduced to a single differential equation by the following transformation:

$$A_{mn}(t) = K_A W_{mn}(t); \quad B_{mn}(t) = K_B W_{mn}(t), \tag{8}$$

where

$$K_A = \frac{C_{12}C_{23} - C_{13}C_{22}}{C_{11}C_{22} - C_{12}^2}$$

$$K_B = \frac{C_{12}C_{13} - C_{11}C_{23}}{C_{11}C_{22} - C_{12}^2}. \tag{9}$$

Transformations and substitution of eqns (5) reduce the set (6) into the following:

$$\ddot{W}_{mn}(t) + \omega_{mn}^2 W_{mn}(t) = P_{mn}\frac{ab}{uv}\frac{F(t)}{m_1}, \tag{10}$$

where

$$\omega_{mn}^2 = \frac{C_{13}K_A + C_{23}K_B + C_{33}}{\rho h}$$

are the squared fundamental frequencies of the plate, and m_1 is its mass.

For zero initial displacement and velocity, the solution of eqn (10) is obtained using the convolution integral:

$$W_{mn}(t) = \frac{P_{mn}ab}{uvm_1\omega_{mn}}\int_0^t F(\tau)\sin\omega_{mn}(t-\tau)\,\mathrm{d}\tau. \tag{11}$$

The deflection of the plate at any point is given by

$$w(x, y, t) = \frac{ab}{uvm_1}\sum_m\sum_n \frac{P_{mn}}{\omega_{mn}}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\int_0^t F(\tau)\sin\omega_{mn}(t-\tau)\,\mathrm{d}\tau. \tag{12}$$

IMPACT DYNAMICS

The response of a plate to impact by a foreign object may be computed from the transient response, eqn (12), by computing the impact force from the deceleration of the impactor mass. The integral equilibrium equation between the impactor and the plate during contact is given by

$$w\left(\frac{a}{2}, \frac{b}{2}, t\right) = V_0 t - \frac{1}{m_2}\int_0^t F(\tau)(t-\tau)\,\mathrm{d}\tau - \frac{F(t)}{K_2}, \tag{13}$$

where V_0 is the initial velocity of the impactor with mass m_2, $w\ (a/2,\ b/2,\ t)$ is the lateral deflection of the plate evaluated at the impact point as a function of time, $F(t)$ is the impact force and K_2 is the linearized contact area stiffness.

As mentioned in the introduction, the contact problem must be linearized in order to be treated analytically using the present approach. This involves both replacing the nonlinear contact stiffness with an appropriate linearized value, as well as approximating the time varying contact area by a constant area. Further discussion on these approximations is given subsequently.

Combining eqns (12) and (13) yields:

$$V_0 t - \frac{1}{m_2}\int_0^t F(\tau)(t-\tau)\,\mathrm{d}\tau - \frac{F(\tau)}{K_2} = \frac{ab}{uvm_1}\sum_m\sum_n \frac{P_{mn}}{\omega_{mn}}\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}\int_0^t F(\tau)\sin\omega_{mn}(t-\tau)\,\mathrm{d}\tau. \tag{14}$$

Taking the Laplace Transform of eqn (14) and after some rearranging yields:

$$F(s) = \frac{m_2 V_0}{1 + \dfrac{m_2}{K_2}s^2 + \dfrac{ab}{uv}\dfrac{m_2}{m_1}\sum_m\sum_n P_{mn}\sin\dfrac{m\pi}{2}\sin\dfrac{n\pi}{2}\dfrac{s^2}{s^2+\omega_{mn}^2}}. \tag{15}$$

Using the inverse theorem of the Laplace Transforms and following the same procedure as Christoforou and Swanson (1988) yields:

$$F(t) = \sum_i F_i \sin\omega_i t, \tag{16}$$

where ω_i are the response frequencies or poles of expression (15) and

$$F_i = \frac{m_2 V_0}{\omega_i\left[\dfrac{m_2}{K_2} + \dfrac{ab}{uv}\dfrac{m_2}{m_1}\sum_m\sum_n P_{mn}\sin\dfrac{m\pi}{2}\sin\dfrac{n\pi}{2}\dfrac{\omega_{mn}^2}{(\omega_{mn}^2-\omega_i^2)^2}\right]}. \tag{17}$$

In the limiting case where the plate is very thick and the response is dominated by the contact deformation effects, eqn (15) becomes

$$F(s) = m_2 V_0 \frac{\omega_c^2}{s^2+\omega_c^2}, \tag{18}$$

and eqn (16) becomes

$$F(t) = m_2 V_0 \omega_c \sin\omega_c t, \tag{19}$$

where $\omega_c^2 = K_2/m_2$ is the frequency associated with contact deformation effects.

The impact response can be determined by combining the Laplace transforms of eqns (11) and (16),

$$W_{mn}(s) = \frac{P_{mn}ab}{uvm_1}\sum_i \frac{F_i\omega_i}{(s^2+\omega_{mn}^2)(s^2+\omega_i^2)}. \tag{20}$$

Using the inverse theorem of the Laplace transform and the same procedure as before yields:

$$W_{mn}(t) = \frac{P_{mn}ab}{m_1 uv} \sum_j \frac{F_j}{\omega_{mn}(\omega_j^2 - \omega_{mn}^2)} (\omega_j \sin \omega_{mn} t - \omega_{mn} \sin \omega_j t). \tag{21}$$

The deflection and strain at any point can then be obtained by substituting (21) into eqns (8) and (3).

RESULTS

A problem of impact of a cross-ply laminated plate by a steel sphere is used to illustrate the solution process. The parameters of the problem are given in Table 1, and the geometry is illustrated in Fig. 1. The convergence of the present method is illustrated in Figs 2 and 3. As would be expected, it takes more terms to converge when strain is being computed, relative to either force or displacement. The convergence in the calculation for impact force and transverse center displacement is quite rapid. Figure 2 shows that a reasonable solution is obtained with as few as five terms, and convergence is demonstrated with 25 terms. The calculation for strain takes significantly more terms, but the difference obtained in the results is not significant beyond 75 terms in each summation, and for practical purposes 50 terms could be used. It should be noted that the alternate terms in the series vanish for center impact; the number of terms mentioned here refers to the number of non-vanishing terms.

The above problem has also been examined by Sun and Chen (1985), and Wu (1986). A comparison of the computed contact force and plate center transverse displacement with

Table 1. Values used in example impact problem

Laminated plate	
Simply supported	
Length = width = 200 mm	
Layup = [0/90/0/90/0] s	
Ply thickness = 0.269 mm	
Material properties	
E_{11} = 141.2 GPa	E_{22} = 9.72 GPa
G_{12} = 5.53 GPa	G_{23} = 3.74 GPa
ν_{12} = 0.30	ν_{23} = 0.30
ρ = 1536 Kg m^{-3}	
Impactor	
12.7 mm diameter steel sphere (8.40 gm)	
Impact velocity = 3.00 m s^{-1}	

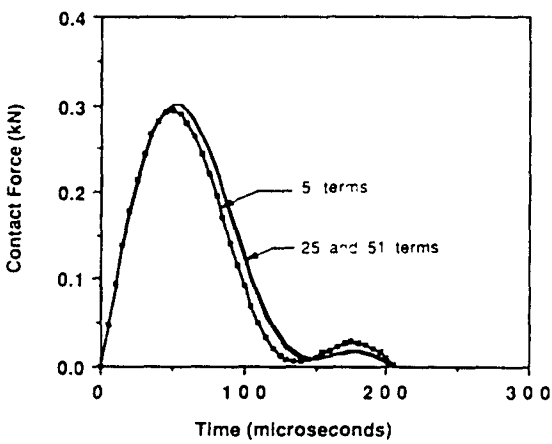

Fig. 2. Convergence of solution for contact force in example impact problem.

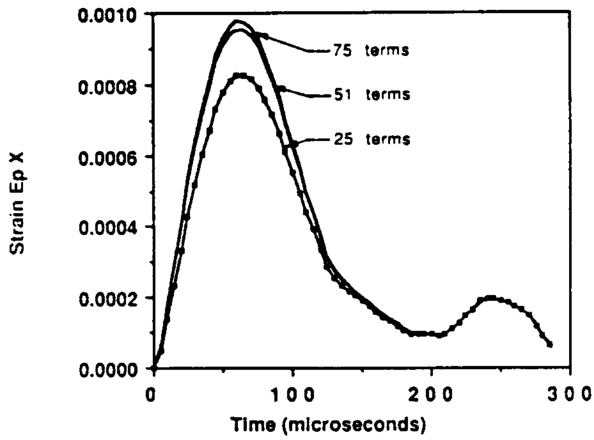

Fig. 3. Convergence of solution for strain behind the impact point in example impact problem.

the solutions of Sun and Chen and Wu is shown in Figs 4 and 5 respectively. The comparison shown indicates that the present method gives a reasonable match with the finite element calculation of Sun and Chen, and also with the results given by Wu, who also used a finite element method. In detail, the analyses are not for precisely the same problem, in that

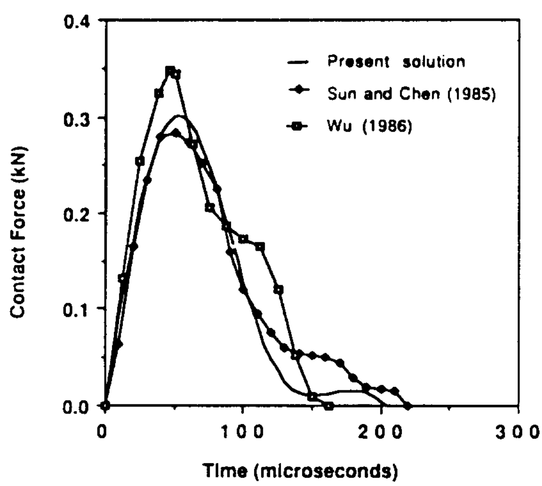

Fig. 4. Comparison of contact force with Sun and Chen (1985) and Wu (1986).

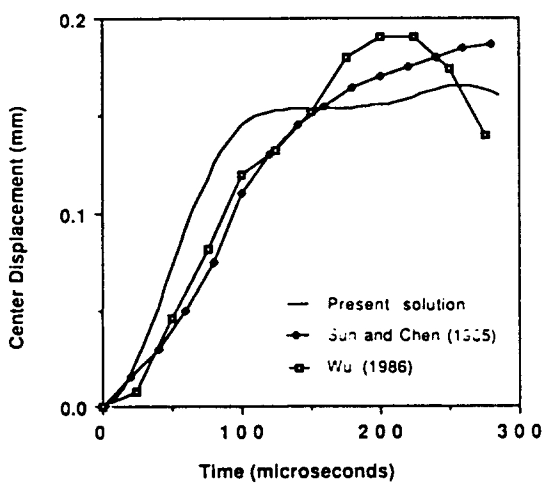

Fig. 5. Comparison of center displacement with Sun and Chen (1985) and Wu (1986).

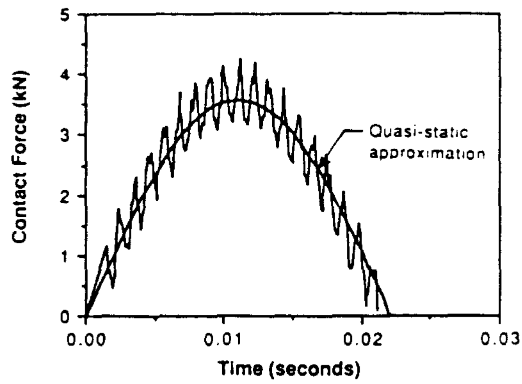

Fig. 6. Comparison with quasi-static approximation for contact force in impact by a heavy mass.

different material properties were used in the two finite element solutions used for comparison. The material properties used at present are essentially the same as those used by Wu (1986), but are about 15% higher than those used by Sun and Chen (1985).

The effect of increasing the mass of the impactor is illustrated in Fig. 6, where the calculated contact force is shown for a problem in which the impact mass has been increased by a factor of 1000. The response depends on both the properties of the plate as well as the impact mass, with the period of the response getting significantly longer with increasing impact mass. In the example shown, the response can be approximated by a simple spring-mass system, with the spring corresponding to the static stiffness of the plate and the mass that of the heavy impactor. This "quasi-static" approximation has been used by a number of investigators.

The effect of increasing the thickness of the plate is illustrated in Figs 7 and 8, where calculated contact force and strain behind the contact point are presented. As would be expected, increasing the thickness of the plate lowers the period of the impact, increases the contact force and decreases the computed strain.

DISCUSSION

The major point of the present paper is to present an analytical method for analysis of the impact problem of orthotropic laminated plates. A solution technique has been presented that uses the usual Fourier series expansions appropriate for simply-supported plates, combined with Laplace transform techniques. The resulting solution gives the contact force, displacements, and stresses and strains within the laminated plate.

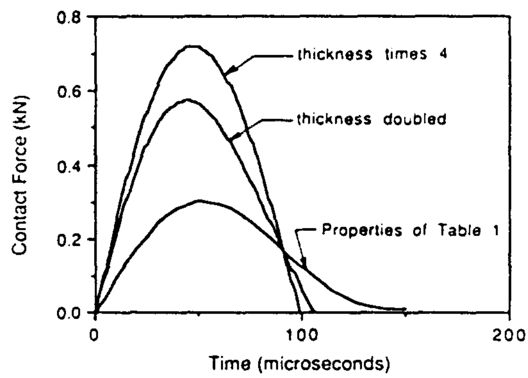

Fig. 7. Effect of plate thickness on contact force in example impact problem.

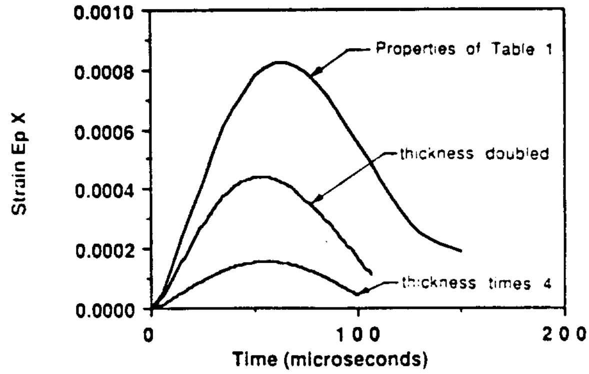

Fig. 8. Effect of plate thickness on strain behind the impact point in example impact problem.

The equations presented above assume that contact is maintained between the impactor and the plate. However this can be easily modified for the case where contact is lost during the impact. The procedure employed is to first calculate the time when the contact force stops being compressive, and then set the contact force to zero at this time. The convolution integral, eqn (11), can then be easily solved by replacing the upper limit of the integral by the time when contact is lost. This solution will hold until contact is again made. A solution to the second impact can be obtained by using the procedures given above, but using the conditions at the time of the second impact as initial conditions for this second impact. This procedure can be continued for subsequent impacts.

A drawback of the present (analytical) solution to the impact problem is that the nonlinear contact force displacement relationship must be approximated by a linearized relationship. The procedure used at present was to base the linear contact stiffness as well as the size of the contact area on values estimated from the static contact loading relations, at an estimated value of peak force for the impact problem. An iteration of solution may be required to obtain an estimate of the peak force. The shape of the contact area and the contact pressure distribution were also simplified, being taken as a square area and a uniform pressure, respectively. These approximations are not significant (within the context of plate equations) as long as the contact area is relatively small.

For problems involving relatively flexible plates, the solution is not particularly sensitive to the numerical value used for the contact stiffness. The linearized contact relationship will thus give good answers in this situation. This can easily be checked by varying the linear contact stiffness used. If a strong dependence on the value used is observed, then a nonlinear numerical solution may be required. However, as mentioned previously by Qian and Swanson (1989), many practical situations involving impact of composite plates do not depend strongly on the contact stiffness value. Neglecting the contact deformation altogether in the present example problem only changed the computed peak force by 10%.

Perhaps the main contribution of an analytical method is that it provides a benchmark for numerical solutions, as the nonlinear numerical solutions may be appropriate in order to include more realism into the impact problem, for example with respect to material damage, more complicated geometries, etc. An example of this type of calibration of a numerical approach may be seen in a comparison of Figs 4 and 5, where the present solution is compared with the finite element solutions of Sun and Chen (1985) and Wu (1986).

As a final point, it may be remarked that it is straightforward to program the present solution for computation. As described above, the method involves finding the roots of a polynomial expression in the square of the Laplace transform parameters s. The response frequencies during the impact are then simply related to the roots by $\omega = (-s^2)^{1/2}$. Any normal root-finding method can be used to find these roots numerically. The procedure used at present was to first calculate the natural frequencies of the plate, which are needed anyway, arrange these natural frequencies in increasing order, and then evaluate the polynomial at intervals between the natural frequencies (between zero and the lowest natural frequency for the first interval). A change in sign of the polynomial indicates the presence of a root; the subinterval containing the root can then be evaluated to determine the precise value of the root and thus the response frequency. We used the bisection method as a "foolproof" brute force technique.

SUMMARY AND CONCLUSIONS

An analytical technique has been presented for impact of laminated composite plates by a rigid impactor. The solution is based on the usual Fourier series expansion for simply-supported plates, combined with Laplace transform techniques to solve the impact problem. The solution gives the impact force and plate displacements, strains and stresses as a function of time. The contact force–displacement relationship can be included in a linearized form.

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